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Article

Numerical solution of age-structured population models using block pulse functions

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Abstract: This paper presents a numerical method for solving a nonlinear age-structured population model based on a set of piecewise constant orthogonal functions. The block-pulse functions (BPFs) method is applied to determine the numerical solution of a non-classic type of partial differential equation with an integral boundary condition. BPFs duo to the simple structure can efficiently approximate the solution of systems with local or non-local boundary conditions. Numerical results reveal the accuracy of the proposed method even for the long term simulations.

Keywords: nonlinear age-structured population model; block pulse functions; integral boundary condition

MSC: 35Q92; 65Nxx; 94A11; 34K28.

1. Introduction

Individuals in a structured population are distinguished by age, size, maturity or some other individual physical characteristics. The main assumption when modeling the evolution of such a population is that the structure of the population with respect to these individual physical characteristics at a given time, and possibly some environmental input as time evolves, completely determines the dynamical behavior of the population. Mathematical models describing this evolution have attracted a considerable amount of interest among researchers as a tool for modeling the interaction of different population communities in such diverse fields as demography, epidemiology, ecology, cell kinetics, tumor growth, etc [1].

In this paper, we will consider the following form of the nonlinear age-structured population model

$$\frac{\partial u(t,x)}{\partial t}+\frac{\partial u(t,x)}{\partial x}=-[d_1(x)+d_2(x)U(t)]u(t,x), \qquad 0\le t, \quad 0\le x<A, \tag{1}$$

$$u(0,x)=u_0(x), \qquad 0\le x<A, \tag{2}$$

$$u(t,0)=\int_a^A [b_1(s)-b_2(s)U(t)]u(t,s)ds, \qquad 0\le t, \tag{3}$$

$$U(t)=\int_0^A u(t,s)ds, \qquad 0\le t, \tag{4}$$

where t,x denote time and age, respectively, $U(t)$ is the total population number at time t, $u(t,x)$ is the age-specific density of individuals of age x at time t, which means that $\int_a^{a+\triangle a} u(t,s)ds$ gives the number of individuals that have age between a and $a+\triangle a$ at time t, $d_1(x)$ is the natural death rate (without considering the competition), $d_2(x)U(t)$ is the increase of death rate considering competition, $b_1(t)$ is the natural fertility rate (without considering competition), $b_2(x)U(t)$ is the decrease of fertility rate considering competition, a denotes the lowest age when an individual can bear, and A is the maximum age that an individual of the population may reach.

Two of the main objectives driving the need for numerical methods are, first, the need to make projections about population growth for the future, usually for periods of 10–50 years. Secondly, there is the theoretical interest in long-term simulations for the purpose of analyzing trends under different scenarios. This is an important aspect of population models in theoretical biology.

Solving the nonlinear systems with an integral boundary condition certainly needs the efficient numerical methods which is also reliable on the large area. Recently, age-structured population models has been solved by many different numerical and analytical methods [1–10]. This study implements the block pulse function method to approximate the solution of this non-classic system. The simple structure and orthogonality of BPFs make them popular in numerical analysis. These functions have been applied several times to solve some different systems arising in engineering and applied sciences [11–25].

The paper is organized as follows: In Section 2, block pulse functions and some properties required for our subsequent development are briefly described. In Section 3, BPFs method associated with collocation scheme is stated. This section explains the algorithm for reduction of the partial integro-differential equation to an algebraic system of equation invoking BPFs. In Section 4, some numerical examples are illustrated which confirm the simplicity, efficiency and reliability of the method. Also, some other features of the presented method and the numerical results are discussed in the conclusion section.

2. Block pulse functions

An m-set of BPFs over the interval $t\in[0,1)$ is defined as follows

$$\phi_i(t)=\begin{cases}1, & t\in[\frac{i}{m},\frac{i+1}{m}),\\ 0, & \text{otherwise},\end{cases} \tag{5}$$

where $i=0,1,\dots,m-1$ is the translation parameter and $\phi_i(t)$ is called ith BPF. More generally, we define the BPFs over the interval $[a,b)$ as

$$\phi_i(t)=\begin{cases}1, & t\in[a+\frac{i(b-a)}{m},a+\frac{(i+1)(b-a)}{m}),\\ 0, & \text{otherwise},\end{cases} \tag{6}$$

for $i = 0,1, \dots, m$.

There are some properties for BPFs which make them popular for approximation such as orthogonality, disjointness and completeness (See [14]). Since the sequence $\{\phi_i(t)\}_{i=0}^{\infty}$ is a complete orthogonal system in $L^2[0,1)$, for $f \in L^2[0,1)$, the series $\sum_i \langle f, \phi_i(t)\rangle \phi_i(t)$ converges uniformly to f where

$$\langle f, \phi_i \rangle = \int_0^1 f(t)\phi_i(t)dt. \tag{7}$$

A function $f(t)$ over the interval $[0,1)$, can be expanded in a BPFs series with an infinite number of terms as

$$f(t) = \sum_{i=0}^{\infty} c_i \phi_i(t), \;\; t \in [0,1), \tag{8}$$

where the coefficients are calculated as follows

$$c_i = m \int_0^1 f(t)\phi_i(t)dt, \;\; i = 0,1, \dots. \tag{9}$$

In fact, the series expansion (8) contains an infinite number of terms for smooth $f(t)$. If $f(t)$ is a piecewise constant or may be approximated as a piecewise constant, the sum in (8) will be terminated after m terms, that is

$$f(t) \simeq \sum_{i=0}^{m-1} c_i \phi_i(t) = C_m^T \Phi_m(t), \;\; t \in [0,1), \tag{10}$$

where

$$C_m = [c_0, c_1, \dots, c_{m-1}]^T \quad \text{and} \quad \Phi_m(t) = [\phi_0(t), \phi_1(t), \dots, \phi_{m-1}(t)]^T. \tag{11}$$

Also, the collocation points can be defined in the following form

$$\tau_l = \frac{2l-1}{2m}, \;\; l = 1,2, \dots, m, \tag{12}$$

which also can be chosen non-uniformly. Now, substituting the collocation points in (10) leads to

$$f(\tau_l) \simeq \sum_{i=0}^{m-1} c_i \phi_i(\tau_l) = C_m^T \Phi_m(\tau_l), \qquad l = 1,2, \dots, m. \tag{13}$$

The above equations can be rewritten in the following matrix form

$$F^T = C_m^T \Phi_m, \tag{14}$$

where

$$F^T = [f(\tau_0), f(\tau_1), \dots, f(\tau_{m-1})]^T, \tag{15}$$

and Φ_m is the BPF matrix of order m defined by

$$\Phi_m = \mathrm{I_m} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{16}$$

Theorem 2.1. *[16,24] Suppose that* $f(t)$ *is an arbitrary real bounded function, which is square integrable in the interval* $[0,1)$*, and*

$$e_m(t) = f(t) - \sum_{i=0}^{m-1} c_i \phi_i(t), \;\; t \in [0,1). \tag{17}$$

Then

$$\parallel e_m(t) \parallel_2 \leq \frac{1}{2\sqrt{3}m} \sup_{t\in[0,1)} |f'(t)|. \tag{18}$$

According to Theorem 2.1, we see that one can obtain an appropriate approximation of the given function using BPFs if m is large enough. Furthermore, since the integration of BPFs has an important role in the approximation of the differential terms, we calculate the integral of (5) in the following form

$$\varphi_i(t) = \int_0^t \phi_i(\tau)\mathrm{d}\tau = \begin{cases} 0, & t \in \left[0, \frac{i}{m}\right), \\ t - \frac{i}{m}, & t \in \left[\frac{i}{m}, \frac{i+1}{m}\right), \\ \frac{1}{m}, & t \in \left[\frac{i+1}{m}, 1\right). \end{cases} \tag{19}$$

Similarly, the function also can be approximated using φ_i as follows

$$f(t) \simeq \sum_{i=0}^{m-1} c_i \varphi_i(t) = C_m^T \Psi_m(t), \;\; t \in [0,1), \tag{20}$$

where

$$\Psi_m(t) = [\varphi_0(t), \varphi_1(t), \dots, \varphi_{m-1}(t)]^T. \tag{21}$$

Substituting the collocation points (12) in (20) yields the following linear system of algebraic equations

$$f(\tau_l) \simeq \sum_{i=0}^{m-1} c_i \varphi_i(\tau_l) = C_m^T \Psi_m(\tau_l), \qquad l = 1,2,\dots,m, \tag{22}$$

which can be rewritten in the following matrix form

$$F^T = C_m^T \Psi_m, \tag{23}$$

where Ψ_m is the BPF integral matrix defined by

$$\Psi_m = \frac{1}{2m}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{24}$$

A function $f(t,x) \in L^2([0,1) \times [0,1)$ can be expanded in BPFs as

$$f(t,x) \simeq \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} f_{kl}\phi_k(t)\bar{\phi}_l(x) = \Phi_{m_1}^T(t) F \bar{\Phi}_{m_2}(x), \tag{25}$$

where $\Phi_{m_1}(t)$ and $\bar{\Phi}_{m_2}(x)$ are m_1-component and m_2-component BPF vectors, respectively, and F is the $m_1 \times m_2$ block pulse coefficient matrix where f_{kl} is given by

$$f_{kl} = m_1 m_2 \int_0^1 \int_0^1 f(t,x)\phi_k(t)\bar{\phi}_l(x), \qquad k = 0,1,\dots,m_1-1, l = 0,1,\dots,m_2-1. \tag{26}$$

Theorem 2.2. *[24] Suppose that* $D = [0,1) \times [0,1)$, $f(t,x) \in L^2(D)$ *and* $e_m(t,x) = f(t,x) - f_m(t,x)$ *which* $f_m(t,x) = \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} f_{kl}\phi_k(t)\phi_l(x)$ *is the block pulse series of* $f(t,x)$. *Then,*

$$\| e_m(t,x) \|_2 \leq \frac{1}{2\sqrt{3}m}\left(\sup_{(t,x)\in D} |f_t(t,x)|^2 + \sup_{(t,x)\in D} |f_x(t,x)|^2 \right)^{\frac{1}{2}}. \tag{27}$$

Regarding (16) and (24), we observe that the linear system of equations derived by BPFs is considerably sparse. Consequently, BPFs are adequate candidate for the interpolation of functions even for problems in higher dimensions. **Figure 1** illustrates that the interpolants become more smooth by increasing the number of the basis functions.

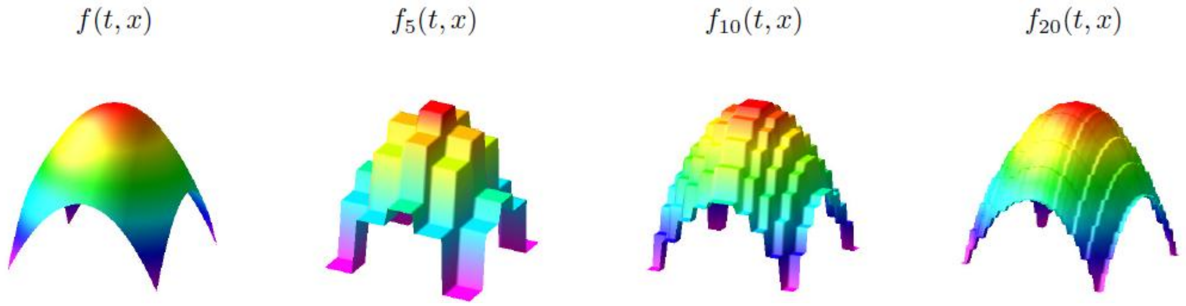

Figure 1. The exact and approximation of surface $f(t,x) = 25 - (x-5)^2 - (t-5)^2$ for different number of block pulse functions.

3. Approximation method

To approximate the solution of the system defined in (1)–(4) which is a first-order partial differential equation, and overcome the discontinuity problem originated from the structure of the block pulse functions, we approximate the second order derivative of the unknown function with respect to x and t as follows

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\phi}_k(t)\hat{\phi}_l(x), \qquad (t,x)\in[0,T)\times[0,A), \tag{28}$$

where $\tilde{\phi}_k(t)$ and $\hat{\phi}_l(x)$ are the BPFs defined in (6) on $[0,T)$ and $[0,A)$ respectively. Note that the notations $\tilde{\phi}$ and $\hat{\phi}$ distinguish the difference between BPFs defined on two different intervals $[0,T)$ and $[0,A)$. Integration of this equation with respect to t and regard to x yields

$$\frac{\partial u(t,x)}{\partial x} = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\varphi}_k(t)\hat{\phi}_l(x) + \frac{\partial u(0,x)}{\partial x}, \tag{29}$$

$$\frac{\partial u(t,x)}{\partial t} = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\phi}_k(t)\,\hat{\varphi}_l(x) + \frac{\partial u(t,0)}{\partial t}, \tag{30}$$

$$u(t,x) = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\varphi}_k(t)\,\hat{\varphi}_l(x) + u(t,0) + u(0,x) - u(0,0), \tag{31}$$

where

$$\tilde{\varphi}_k(t) = \int_0^t \tilde{\phi}_k(\tau)\mathrm{d}\tau, \qquad k=0,1,2,\ldots,m_1-1, \quad \tau,t\in[0,T) \tag{32}$$

$$\hat{\varphi}_l(x) = \int_0^x \hat{\phi}_l(s)\mathrm{d}s, \qquad l=0,1,2,\ldots,m_2-1, \quad s,x\in[0,A). \tag{33}$$

Now we can approximate integral term $U(t)$ in (4) in the following form

$$U(t) = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\varphi}_k(t)\,\zeta_l + \int_0^A u_0(s)\mathrm{d}s + A[u(t,0)-u_{00}], \tag{34}$$

where

$$\zeta_l = \int_0^A\int_0^s \hat{\phi}_l(\rho)\mathrm{d}\rho\mathrm{d}s = \left(\frac{2(m_2-l)-1}{2m_2}\right)A^2, \quad l=0,1,\ldots,m_2, \quad \text{and} \quad u_{00}=u(0,0). \tag{35}$$

Moreover, we set

$$\frac{\partial u(t,0)}{\partial t} = \sum_{k=0}^{m_1-1} c_k\tilde{\phi}_k(t),$$

then

$$u(t,0) = \sum_{k=0}^{m_1-1} c_k\tilde{\varphi}_k(t) + u_{00}, \qquad t\in[0,T), \tag{36}$$

which is compatible with the other stated assumptions. Henceforth, we can rewrite (30), (31) and (34) as follows

$$\frac{\partial u(t,x)}{\partial t} = \sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1} C_{kl}\tilde{\phi}_k(t)\,\hat{\varphi}_l(x) + \sum_{k=0}^{m_1-1} c_k\tilde{\phi}_k(t), \tag{37}$$

$$u(t,x) = \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} C_{kl}\tilde{\varphi}_k(t)\, \hat{\varphi}_l(x) + \sum_{k=0}^{m_1-1} c_k\tilde{\varphi}_k(t) + u_0(x), \tag{38}$$

$$U(t) = \sum_{k=0}^{m_1-1} \sum_{l=0}^{m_2-1} C_{kl}\tilde{\varphi}_k(t)\, \zeta_l + \int_0^A u_0(s)\mathrm{d}s + A\left[\sum_{k=0}^{m_1-1} c_k\tilde{\varphi}_k(t)\right]. \tag{39}$$

We see that the assumed approximation terms include $m_1(m_2+1)$ unknown coefficients

$$C_{kl} \quad \text{and} \quad c_k, \qquad k=0,\dots,m_1-1, \quad l=0,\dots,m_2-1.$$

Now let $R(t,x)$ be the residual function of (1)

$$R(t,x) = \frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} + [d_1(x)+d_2(x)U(x)]u(t,x), \qquad 0\le t\le T, \quad 0\le x<A, \tag{40}$$

which can be reduced by (29), (37), (38) and (39). We choose the following mesh points defined in (12) for spatial and temporal intervals

$$t_i = T\times\frac{2i-1}{2m_1}, \qquad i=1,2,\dots,m_1, \tag{41}$$

$$x_j = A\times\frac{2j-1}{2m_2}, \qquad j=1,2,\dots,m_2. \tag{42}$$

Now we substitute the collocation points in (40) to reduce the partial integro-differential Equations (1)–(4) into the following system of algebraic equations

$$R(t_i,x_j)=0, \qquad i=1,2,\dots,m_1, \;\; j=1,2,\dots,m_2. \tag{43}$$

Substitution of these collocation points into the residual function provide an $m_1\times m_2$ system of equations where still m_1 conditions are needed to find a unique solution. Hence, we substitute t_i, $i=1,2,\dots,m_1$, into (3) which gives other m_1 equations as follows

$$\begin{aligned} u(t_i,0) &= \int_a^A [b_1(s)-b_2(s)U(t_i)]\, u(t_i,s)\mathrm{d}s, \\ &= \frac{A}{m_2}\sum_{j=l}^{m_2} [b_1(x_j)-b_2(x_j)U(t_i)]\, u(t_i,x_j), \qquad i=1,2,\dots,m_1, \end{aligned} \tag{44}$$

where x_l shows the first collocation points defined in (42) larger than a and the integral is computed by the midpoint rule [9]. Also, $u(t_i,0)$, $U(t_i)$ and $u(t_i,x_j)$ should be calculated based on (36), (38) and (39).

4. Numerical results

In this section, some examples are provided to illustrate the computational efficiency of the method. Recall that BPFs are used for intervals $[0,A)$ and $[0,T)$. According to Theorem 2.1, we can decrease the maximum of absolute error by increasing m_1 and m_2. Note that our results have been computed applying a few number of BPFs in order to investigate the method illustratively. It is worth reminding that sparsity of the achieved system is one of the most important privileges of the method.

Example 1. Consider the following nonlinear age-structured population model [5,8,10]

$$\frac{\partial u(t,x)}{\partial t}+\frac{\partial u(t,x)}{\partial x}=-U(t)u(t,x), \qquad 0\le t, \quad 0\le x<A,$$

$$u(0,x)=\frac{e^{-x}}{2}, \qquad 0\le x<A,$$

$$u(t,0)=U(t), \qquad 0\le t,$$

$$U(t)=\int_0^A u(t,s)ds, \qquad 0\le t,$$

which $A=+\infty$. The above equation has the exact solution

$$u(t,x)=\frac{e^{-x}}{e^{-t}+1}, \quad x\ge 0, \quad t\ge 0.$$

The the approximate and the absolute error functions on set $[0,1]\times[0,10]$ which denotes (1 unit time)×(10 unit age) and on $[0,5]\times[0,50]$ are shown in **Figure 2**. Moreover, **Figure 3**. illustrate the approximated values for collocation points when $m_1=5, m_2=50$. Also, $x\in[0,10]$, $[0,50]$, $[0,100]$ and $t_1,t_2,t_3,t_4,t_5\in[0,1]$ which they are given by (41). The results show the validity and reliability of the method on the large area. However approximation on the extended area with the same number of BPFs increases the magnitude of error but higher accuracy can be provided significantly by increasing the number of collocation points for both spatial and temporal intervals.

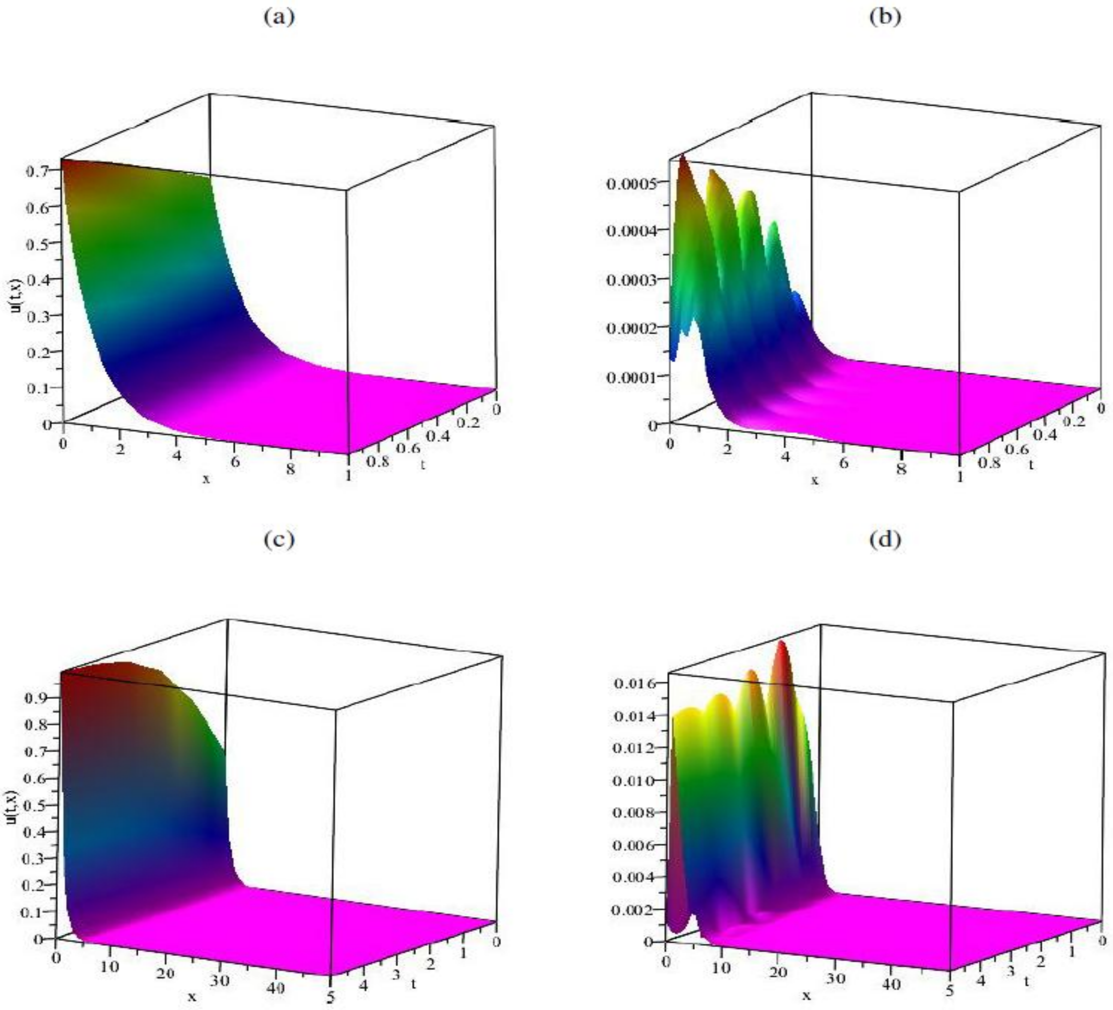

Figure 2. The approximated solution (**a**), (**c**) and the corresponding absolute error function (**b**), (**d**) respectively on $[0,1] \times [0,10]$ and on $[0,5] \times [0,50]$ with $m_1 = 5$ and $m_2 = 50$ for Example 1.

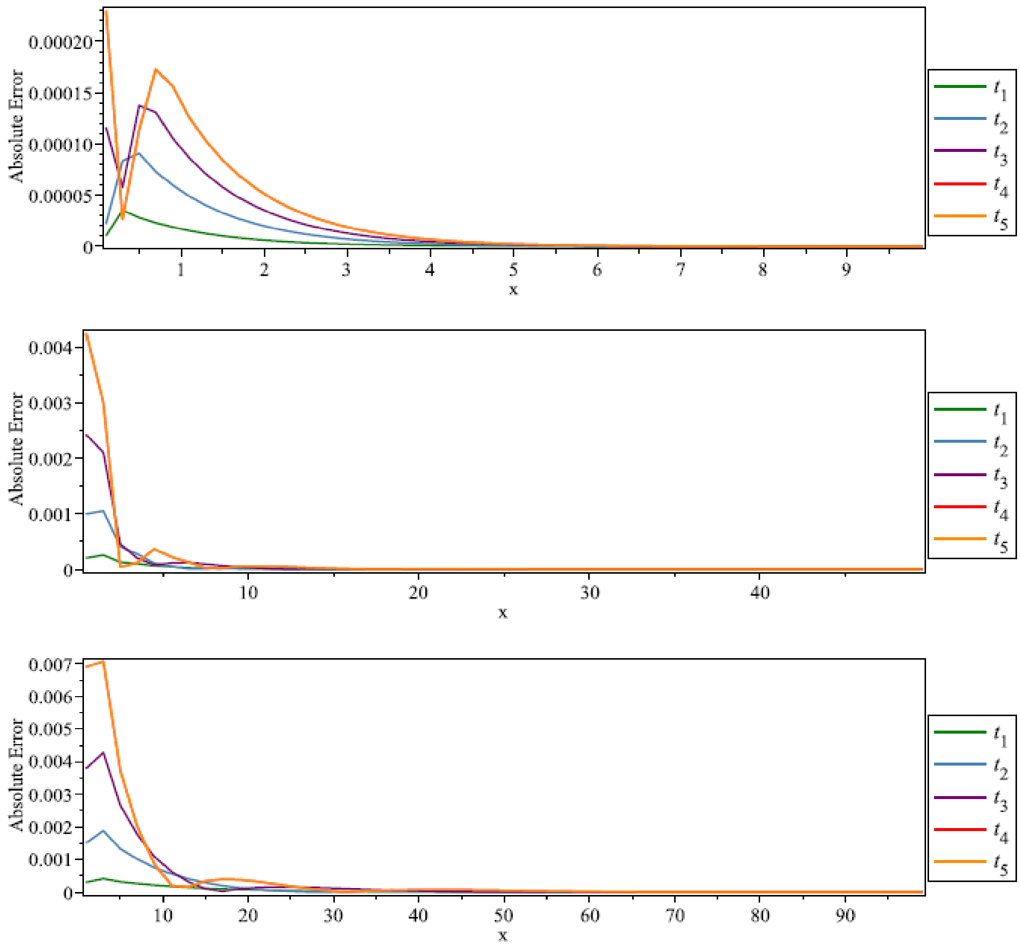

Figure 3. The absolute error value for collocation points with $m_1 = 5$, $m_2 = 50$, $x \in [0,10]$, $[0,50]$, $[0,100]$ and $t_1, t_2, t_3, t_4, t_5 \in [0,1]$ which are given by (41). Note that t_4 and t_5 are close to each other in this case.

Example 2. Consider the following nonlinear age-structured population model [8,10]

$$\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} = -(1 + U(t))u(t,x), \qquad 0 \le t, \quad 0 \le x < A,$$

$$u(0,x) = \frac{e^{-x}}{2}, \qquad 0 \le x < A,$$

$$u(t,0) = U(t), \qquad 0 \le t,$$

$$U(t) = \int_0^A u(t,s)ds, \qquad 0 \le t,$$

which $A = +\infty$. This equation has the exact solution

$$u(t,x) = \frac{e^{-x}}{2+t}, \qquad t \ge 0, \quad x \ge 0.$$

The approximated values for the collocation points (embedded on the exact solution), the corresponding absolute errors, the approximated solution and absolute error function all are depicted in **Figure 4**. for the intervals $[0,1] \times [0,10]$ whit $m_1 = 5, m_2 = 50$. Also, **Figure 5**. includes the approximated and the absolute error functions are illustrated on the large area, $[0,5] \times [0,50]$, which implies that the size of area reversely affects on the magnitude of the error but the reliability of the method still is valid.

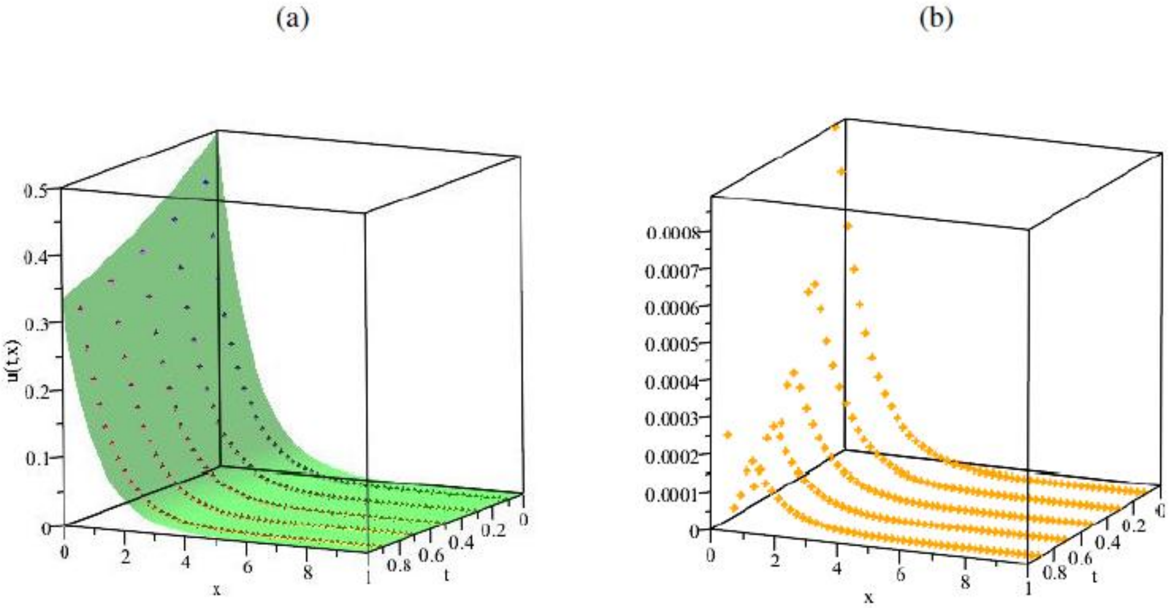

Figure 4. The embedded approximated values for collocation points on the exact solution (**a**) and the corresponding absolute error values (**b**) on $[0,1] \times [0,10]$ with $m_1 = 5$ and $m_2 = 50$ for Example 2.

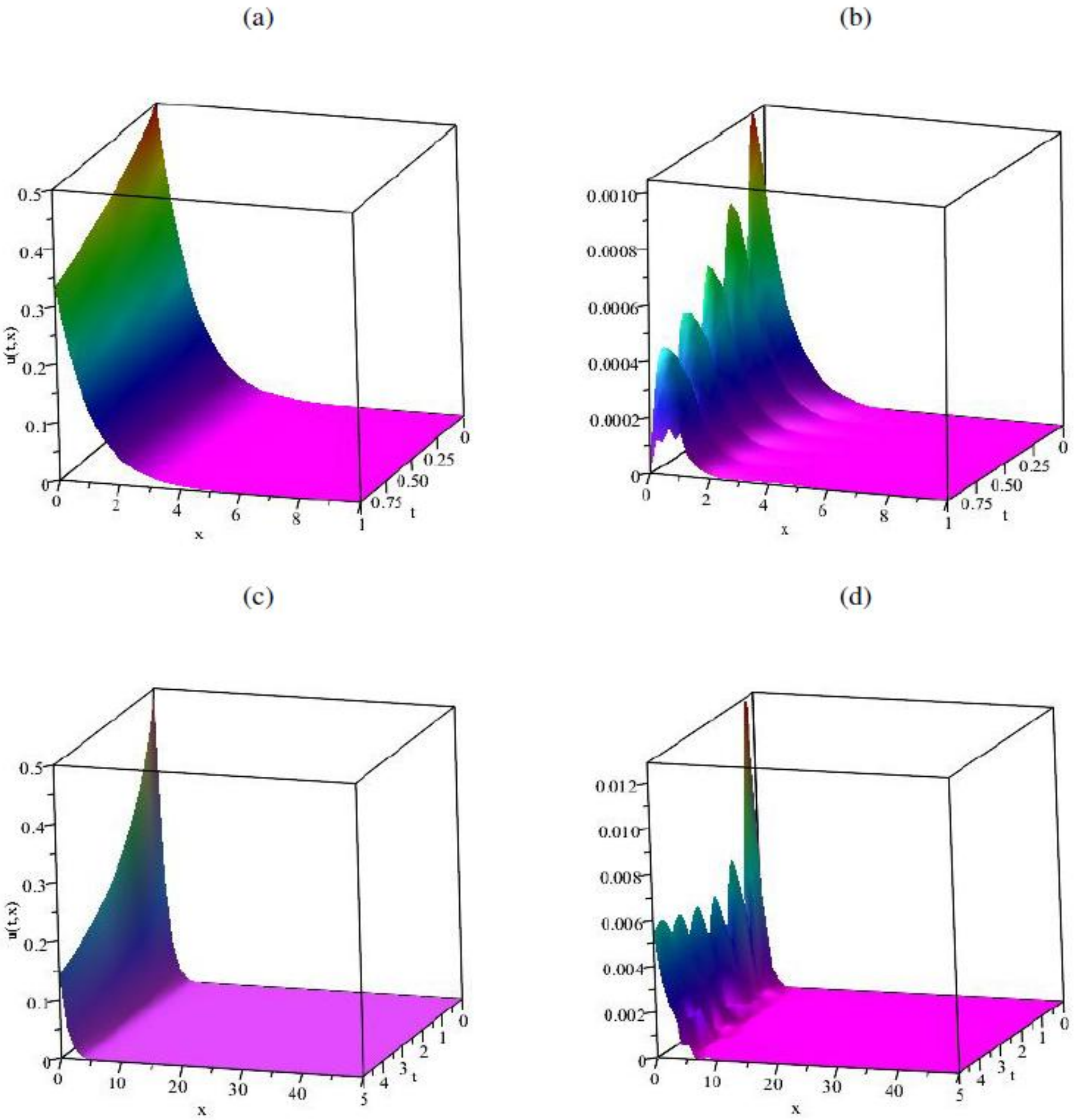

Figure 5. The approximated solution (**a**), (**c**) and the corresponding absolute error function (**b**), (**d**) respectively on $[0,1] \times [0,10]$ and on $[0,5] \times [0,50]$ with $m_1 = 5$ and $m_2 = 50$ for Example 2.

5. Conclusion

In this study, the BPFs method for solving the age-structured population model for different magnitude of domains were investigated. There are some experimental and theoretical analysis about BPFs method which show the efficiency and reliability of the method [11,13–16,22–24]. The numerical experiment for different A and T shows the validity of the method even on the large area which are illustrated in **Figures 2–5**. It means the method can be suitable in the larger scale problems if we choose a large number of the block pulse functions in approximation.

According to the proposed method, we can find $u(t,x)$ continuously for all points of the domain but $\partial u(x,t)/\partial t$ and $\partial u(x,t)/\partial x$ will be discontinuous. Choosing some points of obtained solution and implementing an efficient interpolation method such as interpolation by radial basis functions and rational orthogonal functions, can provide the infinite smooth solution.

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