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Article

Analysis of unsteady heat conduction in regular bodies evolved from uniform surface heat flux: Application of the method of discretization in time coupled with regression analysis

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Abstract: The Method of Discretization in Time (MDT) is a hybrid numerical technique intended to alleviate upfront the computational procedure of time–dependent partial differential equations of parabolic type. The MDT engenders a sequence of adjoint second order ordinary differential equations, wherein the space coordinate is the independent variable while time metamorphosis into an embedded parameter. Fundamentally, the adjoint second order ordinary differential equations are considered of "quasi–stationary" nature. In this work, the MDT is used for the analysis of unsteady heat conduction in regular bodies (large wall, long cylinder and sphere) receiving uniform surface heat flux. In engineering applications, the uniform surface heat flux is customarily provided by electrical heating, radiative heating and pool fire heating. From the outcome of the MDT calculations, it is demonstrated that the approximate, semi–analytical temperature solutions of the first adjoint "quasi–stationary" heat conduction equation based on the first time jump are easily obtainable for the regular bodies. For enhanced accuracy, regression analysis is applied to the deviations of the dimensionless surface temperature varying with the dimensionless time for each regular body.

Keywords: unsteady heat conduction in regular bodies; uniform surface heat flux; method of discretization in time (MDT); adjoint "quasi–stationary" heat conduction equations; regression analysis

1. Introduction

The accurate analysis of unsteady heat conduction in regular bodies (large plate, long cylinder and sphere) involving various heating/cooling conditions is of remarkable importance in engineering applications as cited in heat conduction textbooks by Carslaw and Jaeger [1], Arpaci [2], Luikov [3], Özişik [4] and Poulikakos [5]. For each regular body, the exact analytical solutions of the unsteady heat conduction equation subject to uniform surface heat flux have led to well known infinite series in rectangular, cylindrical and spherical coordinates. Herein, the temperatures are double–valued functions of the space coordinate and time. Beneficially, the infinite series possess intrinsic good convergence and recede to "one term" series when the time is large [1–5]. Regrettably, the infinite series suffer from severe convergence at small time, necessitating many terms to secure adequate accuracy. In fact, the prevalent behavior becomes so abnormal for very short time that the evaluated temperatures may overpredict the initial condition in contraposition with the physics of the problem.

Moreover, the exact analytical solutions of the unsteady heat equation subject to uniform surface heat flux in regular bodies (large plate, long cylinder and sphere) have been considered complex because of two factors: (1) the boundary condition at the body surface is non–homogeneous and (2) no steady–state asymptotic solution exists [1–5]. In engineering applications, the uniform surface heat flux is commonly provided by electrical heating, radiative heating and pool fire heating.

The present study utilizes the Method of Discretization in Time (MDT) to transform the unsteady heat conduction equations for regular bodies (large plate, long cylinder and sphere) into a sequence of "quasi–stationary" heat conduction equations where time becomes an embedded parameter. For enhanced accuracy, the deviations of the temperature–time deviations provided by the MDT are complemented with regression analysis.

The technical paper is divided in six parts. The first part describes the physical system and the problem formulation for the continuous heating of three regular bodies (large plate, long cylinder and sphere) receiving uniform heat flux at the surface. A brief description of the ideas underlying the Method of Discretization in Time (MDT) is delineated in the second part. In the third part, the MDT is applied to the generalized unsteady heat conduction equation for the regular bodies (large plate, long cylinder and sphere) under study. The exact analytical temperature distributions of the three simple bodies as taken from Luikov [3] are disclosed in the fourth part. The semi–analytical temperature distributions for the large plate, long cylinder and sphere produced by the MDT coupled with regression analysis are compared in the fifth part for the three regular bodies.

2. Physical system and mathematical formulation

2.1. Physical system

Consideration is given to unsteady heat conduction in regular bodies (large wall of thickness $2L$, long cylinder of radius R and sphere of radius R). The initial temperature T_{in} in the regular bodies is uniform throughout. For time $t \geq 0$, a uniform heat flux q_s is imposed on the surface(s) of each regular body. The thermal conductivity k and the thermal diffusivity α of the material are assumed nearly constant in the temperature interval of operation $\Delta T = |T_{in} - T_{fin}|$.

2.2. Mathematical formulation

The generalized unsteady heat conduction equation in one dimension is

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{n}{x}\frac{\partial T}{\partial x}\right) \quad \text{in } 0 \leq x \leq L_c, \text{for } t > 0 \tag{1}$$

Herein, the geometric parameter n equates to 0 for rectangular, 1 for cylindrical and 2 for spherical coordinate systems. Also, the characteristic length $L_c = L$ for the large wall and $L_c = R$ for the long cylinder and sphere.

The initial condition is

$$T(x, 0) = T_{in} \tag{2a}$$

and the boundary conditions are

$$\frac{\partial T(0,t)}{\partial x} = 0 \tag{2b}$$

$$\frac{\partial T(L_c,t)}{\partial x} = q_s \tag{2c}$$

Equation (**2c**) exposes the dominant boundary condition containing the uniform surface heat flux q_s which can be provided by electric heating (Houston [6]), radiative heating (Modest [7]) and pool fire heating (Gorbett et al. [8]).

Neumann boundary conditions are given by the function $\frac{\partial T}{\partial x} = f(t)$ in the heat conduction textbook by Özişik [4], so that **Equations** (**2b**) and (**2c**) are particular cases. In more detail, **Equation** (**2b**) is homogeneous implying zero heat flux (thermal symmetry) at the center $x = 0$ of each regular body, whereas **Equation** (**2c**) accounting for uniform heat flux at the surfaces $x = L$ and $x = R$ is non–homogeneous for each regular body.

In the theory of partial differential equations (Larsson and Thomee [9]), the set of **Equations** (**1**) and (**2**) constitutes an initial/boundary value problem (IBVP)

2.3. Target temperatures

For situations involving regular bodies receiving uniform heat flux q_s, the primary target temperature is the surface temperature

$$T_s(t) = T(L_C, t) \tag{3}$$

Herein, $T_s(t)$ must remain below the melting temperature T_{melt} of the solid to maintain the regular bodies in the solid state.

The secondary target temperature is the mean temperature $T_m(t)$. Based on the mean value theorem of calculus

$$T_m(t) = \frac{1}{V} \int_V T(x,t)\, dV \tag{4}$$

where V is the volume of the regular body.

2.4. Dimensionless mathematical formulation

Owing to the nature of Neumann boundary conditions, the appropriate dimensionless variables for the space coordinate *x,* time *t* and temperature difference $T - T_{in}$ are

$$X = \frac{x}{L_c}, \qquad \tau = \frac{t}{t_{eq}}, \qquad \Phi = \frac{T - T_{in}}{T_{eq}} \tag{5}$$

where the respective scales are the characteristic length L_c, the equivalent time $t_{eq} = \frac{L_c^2}{\alpha}$ and the equivalent temperature $T_{eq} = \frac{q_s L_c}{k}$. Notice that the characteristic length L_c appears in the three dimensionless variables X, τ and Φ.

Absorbing the selected dimensionless variables X, τ and Φ, the generalized unsteady heat conduction equation (1) is transformed into

$$\frac{\partial \Phi}{\partial \tau} = \frac{\partial^2 \Phi}{\partial X^2} + \frac{n}{X}\frac{\partial \Phi}{\partial X} \quad \text{in } 0 \le X \le 1, \text{for } \tau > 0 \tag{6}$$

Similarly, the initial condition switches to

$$\Phi\,(X, 0) = 0 \tag{7a}$$

and the boundary conditions become

$$\frac{\partial \Phi\ (0,\tau)}{\partial X} = 0 \tag{7b}$$

$$\frac{\partial \Phi\ (1,\tau)}{\partial X} = 1 \tag{7c}$$

3. Conventional finite difference methods

The three most popular finite difference methods for solving partial differential equations of parabolic type are the explicit method, the implicit method and Crank–Nicolson method (Larsson and Thomee [9]). The explicit method is the easiest to implement and the least numerically intensive, but requires to comply with a stability criterion. The implicit method is unconditionally stable and is first order accurate in time and second order accurate in space, but the solution of a large system of algebraic equations at each time step is required. The Crank–Nicolson method is the most accurate, because is unconditionally stable and is second order accurate in space and time. The implicit and the Crank–Nicolson methods involve large systems of algebraic equations at each time step.

4. The method of discretization in time (MDT)

One of the basic principles of applied mathematics is the divide–and–conquer algorithm The divide and conquer (D & C) algorithm is an algorithm design paradigm that solves complex problems by breaking them into smaller, more manageable sub–problems, which either have been solved or have solutions that are easier to obtain (Cormen et al. [10]). The goal of the present study is inspired in the divide–and–conquer algorithm to analyze unsteady heat conduction in simple bodies (large plate, long cylinder and sphere) promoted by uniform surface heat flux.

The Method of Discretization in Time (Rektorys [11]) sometimes called the Transversal Method of Lines (Rothe [12]) embodies a mathematical procedure for transforming a partial differential equation of parabolic type in two independent variables into a sequence of adjoint ordinary differential equations of second order in terms of the independent space variable x, while the other independent variable time t metamrphosis into an embedded parameter, i.e., a time interval Δt. With regards to the dimensionless unsteady heat conduction equation, the MDT procedure requires a computational domain to be formed with a set of transversal lines parallel to the X–coordinate and perpendicular to the τ–coordinate as illustrated in **Figure 1**. For simplicity, the transversal lines are separated by an equal dimensionless time interval $\Delta\tau$.

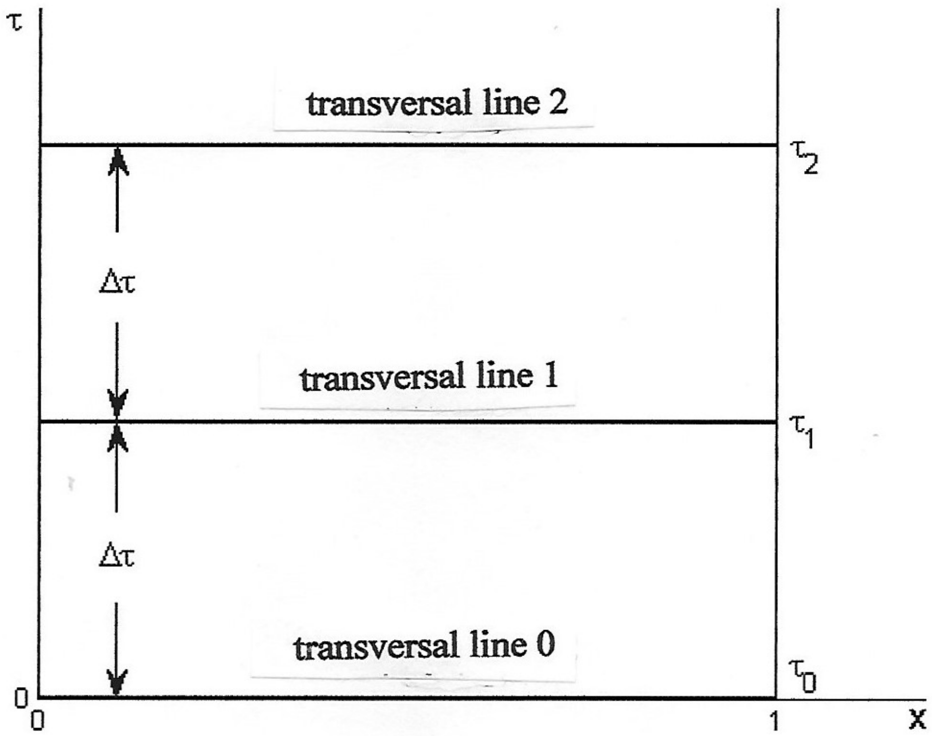

Figure 1. Special computational domain comprising a set of transversal lines for implementing the MDT also called the TMOL.

The underlying idea of the MDT for **Equation** (**6**) is to discretize the first order partial derivative in time $\frac{\partial \Phi}{\partial \tau}$, while leaving the first and second order partial derivatives in space $\frac{\partial \Phi}{\partial X}$ and $\frac{\partial^2 \Phi}{\partial X^2}$ continuous. Correspondingly, the two–point backward finite difference approximation for $\frac{\partial \Phi}{\partial \tau}$ is

$$\left.\frac{\partial \Phi_p}{\partial \tau}\right|_{\tau_P} \approx \frac{\Phi_p - \Phi_{p-1}}{\Delta \tau} + O(\Delta \tau) \qquad (8)$$

$$\text{for } p = 1, 2, 3, \ldots$$

Herein, the appended term $O\,(\Delta\,\tau)$ indicates truncation error of first order (Chapra and Canale [13]).

In view of the foregoing, the substitution of **Equation** (**8**) into **Equation** (**5**) generates a set of adjoint second order ordinary differential equations in the dimensionless space variable X that is specified at successive levels of the dimensionless time τ_p (p = 1, 2, ..., P). That is:

$$\frac{d^2\Phi_p}{dX^2} + \frac{n}{X}\frac{d\Phi_p}{dX} = \frac{\Phi_p}{\Delta\tau} - \frac{\Phi_{p-1}}{\Delta\tau} \text{ in } 0 \leq X \leq 1 \text{ for } p = 1, 2, 3,\ldots, \text{ and constant } \Delta\tau \qquad (9)$$

From **Equation** (**8**), it may be inferred that **Equation** (**9**) revolves around a truncation error of first order (Chapra and Canale [13]).

From a perspective of thermal physics, **Equation** (**9**) representative of the dimensionless temperature variable $\Phi_p : p = 1, 2, 3, \ldots$ may be viewed as a "quasi–

stationary" heat conduction equation operating at a fixed dimensionless time τ_p associated with the transversal line p = 1, 2, 3,...

Likewise, the applicable boundary conditions from **Equation** (**7b**) and (**7c**) are converted into

$$\frac{d\Phi_p(0)}{dX} = 0 \tag{10a}$$

$$\frac{d\Phi_p(1)}{dX} = 1 \tag{10b}$$

for $p = 1, 2, 3, \ldots..$

From a fundamental standpoint, the trio of **Equations** (**9**)**,** (**10a**) and (**10b**) establishes a two point boundary value problem (Zill and Cullen [14]).

Since the ordinary differential equation of second order (9) is linear, it can be integrated with analytical methods to obtain an approximate semi–analytical solution $\Phi_p(X)$ at each transversal line $p = 1, 2, 3, \ldots.$

Beneficially, the main advantage of the hybrid MDT over the conventional finite difference methods (explicit, implicit and Crank–Nicolson) is that the truncation errors are quantified in terms of the dimensionless space variable *X*.

4.1. Implementation of the MDT

To implement the MDT, we commence with the first time jump p = 1 from the transversal line 0 (the initial condition) to the transversal line 1 at a dimensionless time $\tau_1 = \Delta\tau$ in **Figure 1**. Correspondingly, the first adjoint ordinary differential equation of second order is expressed by

$$\frac{d^2\Phi_1}{dX^2} + \frac{n}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = -\frac{\Phi_0}{\Delta\tau} \tag{11}$$

Next, the dimensionless initial condition $\Phi\,(X, 0) = \Phi_0 = 0$ from **Equation** (**10a**) is introduced into **Equation** (**11**). This step gives way to the first dimensionless "quasi–stationary" heat conduction equation:

$$\frac{d^2\Phi_1}{dX^2} + \frac{n}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{12}$$

subject to the boundary conditions

$$\frac{d\Phi_1(0)}{dX} = 0 \tag{13a}$$

$$\frac{d\Phi_1(1)}{dX} = 1 \tag{13b}$$

Herein, the subscript "1" attached to the symbol Φ designates the first dimensionless time jump from the transversal line 0 (the initial condition) to the neighbor transversal line 1.

5. "Quasi–stationary" heat conduction

The ensuing "quasi–stationary" heat conduction equations operating at the dimensionless time τ_1 on the transversal line 1 in the regular bodies are listed next

(1) Large wall

Assigning $n = 0$ to **Equation (12)**, the ODE of second order with constant coefficients is

$$\frac{d^2\Phi_1}{dX^2} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{14}$$

so that **Equations (14), (13a)** and **(13b)** establish a two point boundary value problem (Zill and Cullen [14]).

The general solution of **Equation (14)** absorbing **Equations (13a)** and **(13b)** delivers the particular solution. This gives rise to the dimensionless semi–analytical temperature sub–distribution

$$\Phi_1(X, \Delta\tau) = \left[\frac{\sqrt{\Delta\tau}}{\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] \cosh\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{15}$$

At this point, it is worth remarking that the validity of **Equation (15)** is limited to a small time sub– domain $\Delta\tau$ in conformity with the truncation error of first order $O(\Delta\tau)$ stated in **Equation (8)**.

Evaluating **Equation (15)** at the surface (the dimensionless $X = 1$) delivers the dimensionless semi–analytical surface temperature sub–distribution

$$\Phi_{1,s}(\Delta\tau) = \Phi_1(1, \Delta\tau) = \frac{\sqrt{\Delta\tau}}{\tanh\left(\frac{1}{\sqrt{\Delta\tau}}\right)} \tag{16}$$

which is limited to a small time sub–domain $\Delta\tau$.

In addition, the dimensionless mean temperature distribution from **Equation (4)** is

$$\Phi_{1,m}(\Delta\tau) = \int_0^1 \Phi_1(X, \Delta\tau)\, dX \tag{17}$$

where the dimensionless temperature distribution $\Phi_1(X, \Delta\tau)$ is **Equation (15)**. Alternatively, the dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained exactly from the so–called zero–dimensional heat equation (see **Appendix**). At the end, the resulting dimensionless mean temperature distribution is expressed by the linear algebraic equation with slope one

$$\Phi_{1,m}(\tau) = \Delta\tau \tag{18}$$

Note that this equation is valid in the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics.

(2) Long cylinder

Assigning $n = 1$ to **Equation (12)**, the ODE of second order with constant coefficients is named the modified Bessel equation

$$\frac{d^2\Phi_1}{dX^2} + \frac{1}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{19}$$

Equations (19), **(13a)** and **(13b)** establish a two point boundary value problem (Zill and Cullen [14]).

The general solution of **Equation (19)** along with the boundary conditions in **Equations (13a)** and **(13b)** generates the particular solution that gives rise to the dimensionless semi–analytical temperature sub–distribution

$$\Phi_1(X, \Delta\tau) = \left[\frac{\sqrt{\Delta\tau}}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] I_0\left(\frac{X}{\sqrt{\Delta\tau}}\right) \tag{20}$$

where I_0 is the modified Bessel functions of second kind and order zero and I_1 is the modified Bessel functions of second kind and order one. The validity of **Equation** (**20**) is for a small time sub– domain $\Delta\tau$ in conformity with the truncation error of first order $O(\Delta\tau)$ stated in **Equation** (**8**).

Evaluating **Equation** (**20**) at the surface (the dimensionless $X = 1$), provides the dimensionless semi–analytical surface temperature sub–distribution

$$\Phi_{1,s}(\Delta\tau) = \Phi_1(1, \Delta\tau) = \sqrt{\Delta\tau}\left[\frac{I_0\left(\frac{1}{\sqrt{\Delta\tau}}\right)}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] \tag{21}$$

In addition, from **Equation** (**4**) the dimensionless mean temperature distribution is

$$\Phi_{1,m}(\Delta\tau) = 2\int_0^1 \Phi_1(X, \Delta\tau)\, X\, dX \tag{22}$$

where the dimensionless temperature distribution $\Phi_1(X, \Delta\tau)$ is taken from **Equation** (**20**). Alternatively, the dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained exactly from the so–called zero–dimensional heat conduction equation). At the end, the resulting dimensionless mean temperature distribution is expressed by the linear algebraic equation with slope two

$$\Phi_{1,m}(\tau) = 2\Delta\tau \tag{23}$$

Note that this equation is valid in the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics Theory.

(3) Sphere

Equation (**12**) particularized to $n = 2$ furnishes the ODE of second order with variable coefficients

$$\frac{d^2\Phi_1}{dX^2} + \frac{2}{X}\frac{d\Phi_1}{dX} - \frac{\Phi_1}{\Delta\tau} = 0 \tag{24}$$

Equations (**24**), (**13a**) and (**13b**) establish a two point boundary value problem (Zill and Cullen [13]).

The particular solution of **Equation** (**24**) provides the dimensionless semi–analytical temperature sub–distribution

$$\Phi_1(X, \Delta\tau) = \left[\frac{\sqrt{\Delta\tau}}{\cosh\left(\frac{1}{\sqrt{\Delta\tau}}\right) - \sqrt{\Delta\tau}\sinh\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right]\frac{\sinh\left(\frac{X}{\sqrt{\Delta\tau}}\right)}{X} \tag{25}$$

with validity in a small time sub–domain $\Delta\tau$ in conformity with the truncation error of first order $O(\Delta\tau)$ stated in **Equation** (**8**).

From here, the dimensionless semi–analytical surface temperature is

$$\Phi_{1,s}(\Delta\tau) = \Phi_1(1, \Delta\tau) = \frac{\sqrt{\Delta\tau}}{\coth\left(\frac{1}{\sqrt{\Delta\tau}}\right) - \sqrt{\Delta\tau}} \tag{26}$$

In addition, the dimensionless mean temperature distribution from **Equation** (**4**) is

$$\Phi_{1,m}(\Delta\tau) = 3\int_0^1 \Phi_1(X,\Delta\tau)\, X^2\, dX \tag{27}$$

where the dimensionless mean temperature distribution $\Phi_1(X,\Delta\tau)$ is taken from **Equation** (**25**). Here again, the analytical dimensionless mean temperature distribution $\Phi_{1,m}(\Delta\tau)$ can be obtained from the so–called zero–dimensional heat conduction equation). At the end, the resulting dimensionless mean temperature distribution is expressed by the linear algebraic equation with slope three

$$\Phi_{1,m}(\tau) = 3\Delta\tau \tag{28}$$

Note that this equation is valid in the entire time domain $0 < \tau < \infty$ in conformity to Thermodynamics Theory.

6. Exact, analytical surface temperature distributions

Notwithstanding, the most important target temperature in the continuous heating of regular bodies (large wall, long cylinder and sphere) with uniform surface heat flux turns out to be the surface temperature, i.e., the highest temperature in each regular body at any given time. In dimensionless form, that is $\Phi_s(\tau) = \Phi(1,\tau)$.

(1) Large wall

From **Equation** (**A1a**), the dimensionless exact, analytical surface temperature is

$$\Phi_s(\tau) = \Phi(1,\tau) = \tau + \frac{1}{3} - 2\sum_{n=1}^{\infty} (-1)^{\mathrm{n}+1}\left[\frac{\cos(\mu_n)}{\mu_n^2}\right] \exp(-\mu_n^2\,\tau) \tag{29}$$

covering the entire time domain $0 < \tau < \infty$.

Taking the limit $\tau \to \infty$ in the exponential function $\exp(-\mu_n^2\,\tau)$ in **Equation** (**29**) gives way to the asymptotic dimensionless surface temperature sub–distribution

$$\Phi_{s,asy}(\tau) = \tau + \frac{1}{3} \tag{30}$$

(2) Long cylinder

From **Equation** (**A2a**), the dimensionless exact, analytical surface temperature is

$$\Phi_s(\tau) = \Phi(1,\tau) = 2\tau + \frac{1}{4} - 2\sum_{n=1}^{\infty} \left(\frac{1}{\mu_n^2}\right) \exp(-\mu_n^2\,\tau) \tag{31}$$

covering the entire time domain $0 < \tau < \infty$.

Taking the limit $\tau \to \infty$ in the exponential function $\exp(-\mu_n^2\,\tau)$ in **Equation** (**31**) give ways to the asymptotic dimensionless surface temperature sub–distribution

$$\Phi_{s,asy}(\tau) = 2\tau + \frac{1}{4} \tag{32}$$

(3) Sphere

From **Equation** (**A3a**), the dimensionless exact, analytical surface temperature reduces to

$$\Phi_s(\tau) = \Phi(1,\tau) = 3\tau + \frac{1}{5} - 2\sum_{n=1}^{\infty} \left[\frac{\tan(\mu_n)}{\mu_n^3}\right] \exp(-\mu_n^2\,\tau) \tag{33}$$

covering the entire time domain $0 < \tau < \infty$.

Taking the limit $\tau \rightarrow \infty$ in the exponential function $\exp\left(-\mu_n^2\, \tau\right)$ in **Equation (33)** gives way to the asymptotic dimensionless surface temperature sub–distribution

$$\Phi_{s,asy}(\tau) \ = 3\tau + \frac{1}{5} \tag{34}$$

The three infinite series in **Equations (29), (31)** and **(33)** are evaluated numerically with the symbolic algebra software Mathematica [15] accounting for a term–by–term convergence criterion with error size 0.01%.

7. Dimensionless treshold times

First, the dimensionless threshold time τ_{th} for the large wall prevalent between **Equations (29)** and **(30)** is set at τ_{th} = 0.30 in **Figure 2**. Second, the dimensionless threshold time τ_{th} for the long cylinder prevalent between **Equations (31)** and **(32)** is set at τ_{th} = 0.20 in **Figure 3**. Third, the dimensionless threshold time τ_{th} for the sphere prevalent between **Equations (33)** and **(34)** is set at τ_{th} = 0.10 in **Figure 4**.

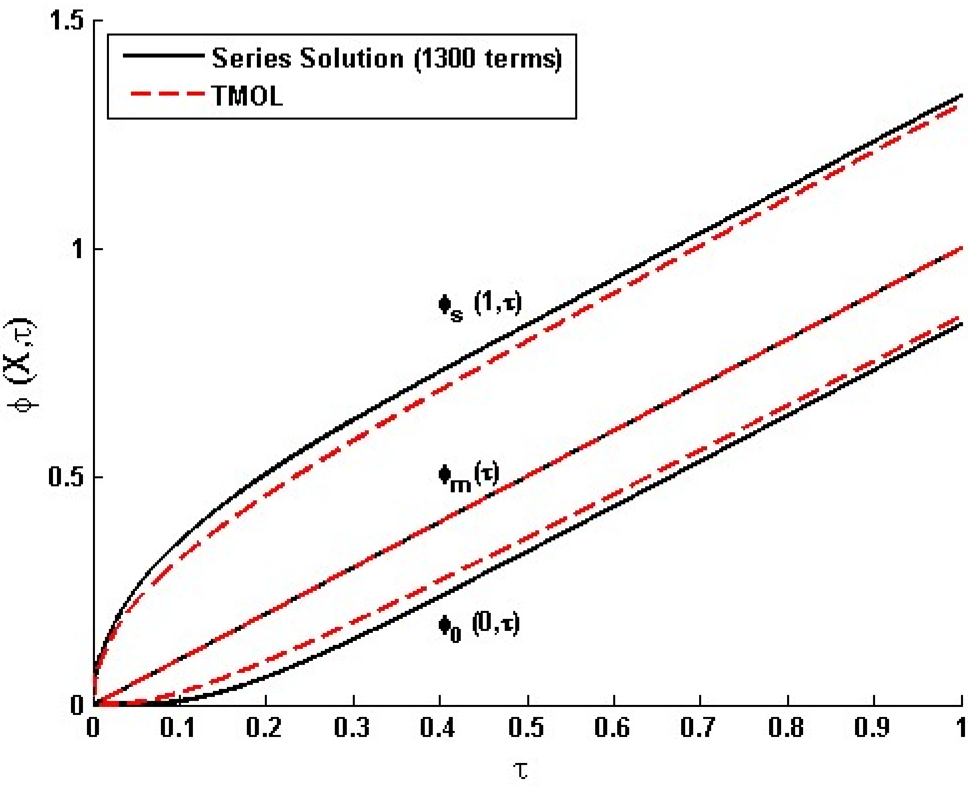

Figure 2. Center, surface and mean dimensionless temperature distributions for a plane wall with the MDT also called the TMOL.

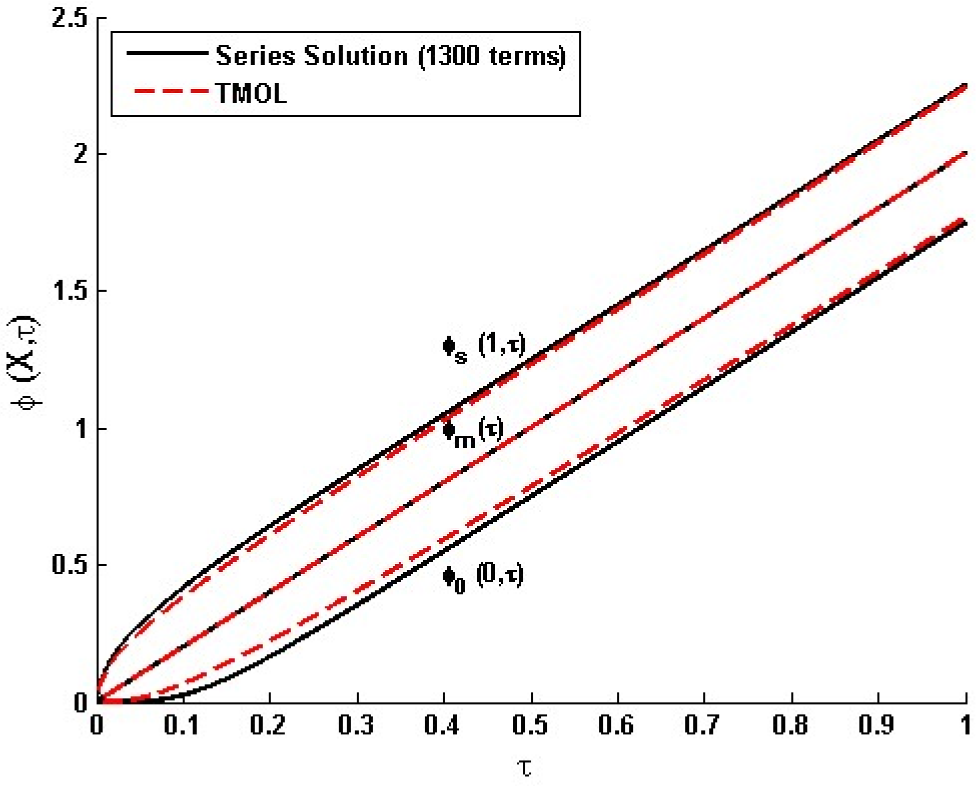

Figure 3. Center, surface and mean dimensionless temperature distributions for a long cylinder with the MDT als called the TMOL.

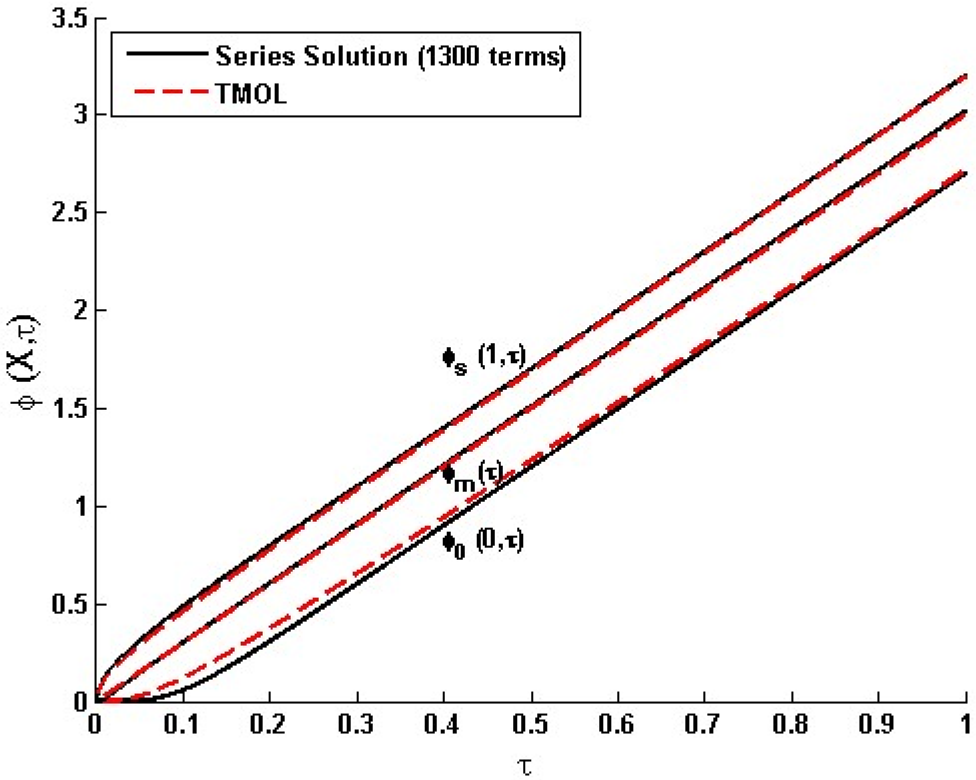

Figure 4. Center, surface and mean dimensionless temperature distributions for a sphere with the MDT also called the TMOL.

8. Avenues to improve the surface temperatures

To improve the accuracy of the surface temperatures delivered by the MDT involving the first jump, there are two possible options.

8.1. Option 1

The adjoint ordinary differential equation of second order and inhomogeneous at the dimensionless time τ_2 should go to two time jumps: first $\tau_1 = \Delta\tau/2$ and second $\tau_2 = \tau_1 + \Delta\tau/2$. Here again, usage is made of the two–point backward finite–difference approximation

$$\left.\frac{\partial z}{\partial t}\right|_{t_2} \approx \frac{z_2 - z_1}{\Delta t} + O(\Delta t) \tag{35}$$

where the term $O(\Delta\tau)$ denotes truncation error of first order (Chapra and Canale [13]). Accordingly, this operation leads to the inhomogeneous, "quasi–stationary" heat equation

$$\frac{d^2\Phi_2}{dX^2} + \frac{n}{X}\frac{d\Phi_2}{dX} - \frac{\Phi_2}{\Delta\tau/2} = -\frac{1}{\Delta\tau/2}\left[\Phi_1\left(X, \Delta\tau/2\right)\right] \tag{36}$$

along with the boundary conditions

$$\frac{d\Phi_2(0)}{dX} = 0 \tag{37a}$$

$$\frac{d\Phi_2(1)}{dX} = 1 \tag{37b}$$

Herein, the term $\frac{1}{\Delta\tau}\left[\Phi_1\left(X, \Delta\tau/2\right)\right]$ on the right hand side of **Equation** (**36**) stands for the dimensionless temperature distribution associated with the first time jump $\Delta\tau/2$.

Here again, **Equations** (**36**)**,** (**37a**) and (**37b**) establish a two point boundary value problem (Zill and Cullen [14]).

The corresponding ordinary differential equations of second order are

(1) Large wall

$$\frac{d^2\Phi_2}{dX^2} - \frac{\Phi_2}{\Delta\tau/2} = -\left[\frac{\sqrt{\Delta\tau/2}}{\sinh\left(\frac{1}{\sqrt{\Delta\tau/2}}\right)}\right]\cosh\left(\frac{X}{\sqrt{\Delta\tau/2}}\right) \tag{38}$$

(2) Long cylinder

$$\frac{d^2\Phi_2}{dX^2} + \frac{1}{X}\frac{d\Phi_2}{dX} - \frac{\Phi_2}{\Delta\tau/2} = -\left[\frac{\sqrt{\Delta\tau/2}}{I_1\left(\frac{1}{\sqrt{\Delta\tau/2}}\right)}\right] I_0\left(\frac{X}{\sqrt{\Delta\tau/2}}\right) \tag{39}$$

(3) Sphere

$$\frac{d^2\Phi_2}{dX^2} + \frac{2}{X}\frac{d\Phi_2}{dX} - \frac{\Phi_2}{\Delta\tau/2} = -\left[\frac{\sqrt{\Delta\tau/2}}{\cosh\left(\frac{1}{\sqrt{\Delta\tau/2}}\right) - \sqrt{\Delta\tau/2}\,\sinh\left(\frac{1}{\sqrt{\Delta\tau/2}}\right)}\right]\frac{\sinh\left(\frac{X}{\sqrt{\Delta\tau/2}}\right)}{X} \tag{40}$$

Inspection of the trio of **Equations** (**38**)–(**40**) demonstrates that Option 1 contains a complex non homogeneous term. Because of this, Option 1 is very laborious and time consuming even with the help of a symbolic algebra software, such as Mathematica [15].

8.2. Option 2

To overcome the difficulties posed by the trio of **Equations** (**38**)–(**40**), a short cut approach is to apply regression analysis to the dimensionless surface temperature deviations $\Delta\, \Phi_s$, which are listed in the fourth columns of **Tables 1–3**. The outcome delivers the following set of dimensionless temperature sub–distributions

(1) Large wall

$$\Phi_{s,ra}(\Delta\tau) = \frac{\sqrt{\Delta\tau}}{\tanh\left(\frac{1}{\sqrt{\Delta\tau}}\right)} + f_{lw}(\Delta\tau) \tag{41}$$

where the subscript "*ra*" attached to the symbol Φ_s denotes regression analysis. The correcting function $f_{lw}(\Delta\tau) = 0.085\,(\Delta\tau)^{0.340}$ displays a high correlation coefficient $R = 0.99$.

(2) Long cylinder

$$\Phi_{s,ra}(\Delta\tau) = \sqrt{\Delta\tau}\left[\frac{I_0\left(\frac{1}{\sqrt{\Delta\tau}}\right)}{I_1\left(\frac{1}{\sqrt{\Delta\tau}}\right)}\right] + f_{lc}(\Delta\tau) \tag{42}$$

where the subscript "*ra*" attached to the symbol Φ_s denotes regression analysis. The correcting function. $f_{lc}(\Delta\tau) = 0.053\,(\Delta\tau)^{0.218}$ displays a high correlation coefficient $R = 0.99$.

(3) Sphere

$$\Phi_{s,ra}(\Delta\tau) = \frac{1}{\frac{1}{\sqrt{\Delta\tau}}\coth\left(\frac{1}{\sqrt{\Delta\tau}}\right) - 1} + f_{sp}(\Delta\tau) \tag{43}$$

where the subscript *"ra"* attached to the symbol Φ_s denotes regression analysis. The correcting function. $f_{sp}(\Delta\tau) = 0.04\,(\Delta\tau)^{0.170}$ displays a high correlation coefficient $R = 0.99$.

Table 1. Numerical values of the dimensionless surface temperature $\Phi_s(\tau)$ for a large wall obtained with the MDT inside the dimensionless time sub – domain $0 < \tau \leq 0.30$.

Dimensionless time τ	**Exact analytical Φ_s from Equation (29)**	**Approximate semi–analytical Φ_s with MDT in Equation (16)**	**Deviation $\Delta\Phi_s$** and (**Error**)	**Asymptotic $\Phi_{s,asy}$ with Equation (30)**	**Deviation $\Delta\Phi_s$** and (**Error**)
0	0	0	0		
0.10	0.35683	0.31736	– 0.0394 (–11.06%)		
0.20	0.50517	0.45755	0.0476 (−9.43%)		
0.3 (Dimensionless threshold time)	0.63379	0.60765	−0.5688 (−4.12%)	0.63333	−0.00046 (−0.07%)
All τ			RMSE =		

8.72%

Table 2. Values of the dimensionless surface temperature $\Phi_s(\tau)$ for a long cylinder obtained with MDT inside the sub – domain $0 < \tau \leq 0.20$.

Dimensionless time τ	**Exact analytical Φ_s from Equation (31)**	**Approximate semi−analytical Φ_s with MDT in Equation (21)**	**Deviation $\Delta\Phi_s$ (error)**	**Asymptotic $\Phi_{s,asy}$ with Equation (32)**	**Deviation $\Delta\Phi_s$ (error)**
0	0	0			
0.05	0.28104	0.25424	−0.0268 (−9.54%)		
0.10	0.41833	0.38503	−0.0333 (−7.96%)		
0.20 (Dimensionless threshold time)	0.64277	0.61014	−0.0326 (−5.08%)	0.65000	0.00723 (1.12%)
			RMSE = 7.44%		

Table 3. Numerical values of the dimensionless surface temperature $\Phi_s(\tau)$ for a sphere obtained with MDT inside sub – domain $0 < \tau \leq 0.10$.

Dimensionless time τ	**Exact analytical Φ_s, Equation (33)**	**Approximate semi−analytical Φ_s with DMT, eq.(26)**	**Deviation $\Delta\Phi_s$ (error)**	**Asymptotic $\Phi_{s,asy}$, Equation (34)**	**Deviation $\Delta\Phi_s$ (error)**
0	0	0			
0.05	0.31217	0.28791	−0.02426 (7.77%)		
0.10 (Dimensionless threshold time)	0.48676	0.46006	−0.02670 (5.49%)	0.50	0.01324 (2.72%)

9. Presentation of results

9.1. MDT

Figures 2–4 illustrate the center, surface and mean dimensionless temperatures $\Phi_o(\tau)$, $\Phi_s(\tau)$ and $\Phi_m(\tau)$ obtained with the MDT along with the exact analytical procedures. **Tables 1–3** lists the evaluated dimensionless surface temperatures varying with the dimensionless time $\Phi_s(\tau)$ in each regular body with the MDT. For the large wall, the root mean squared error is RMSE = 9.86% in association with the sub–domain $0 < \tau \leq \tau_{th} = 0.30$. For the long cylinder, the root mean squared error is RMSE = 7.44% in association with the sub–domain $0 < \tau \leq \tau_{th} = 0.20$. For the sphere, the root mean squared error is RMSE = 6.69% in association with the sub–domain $0 < \tau \leq \tau_{th} = 0.10$.

9.2. MDT coupled with regression analysis

Tables 4–6 summarizes the evaluated dimensionless surface temperatures varying with the dimensionless time $\Phi_{s,ra}(\tau)$ in each regular body determined with the MDT coupled with regression analysis. The root mean square error RMSE associated with the MDT coupled with regression analysis are: (1) RMSE = 0.21% (47 times less than with MDT alone) for the large wall in the sub–domain $0 < \tau \leq \tau_{th} =$

0.30, (2) RMSE = 0.08% (93 times less than with MDT alone) for the long cylinder in the sub–domain $0 < \tau \leq \tau_{th}$ = 0.20 and (3) RMSE = 0.07% (97 times less than with MDT alone) for the sphere in the sub–domain $0 < \tau \leq \tau_{th}$ = 0.10.

Table 4. Numerical values of the dimensionless surface temperature $\Phi_s(\tau)$ for a large wall obtained with the MDT coupled with regression analysis inside to the sub–domain $0 < \tau \leq 0.30$.

Dimensionless time τ	**Exact analytical Φ_s from Equation (29)**	**Semi–analytical $\Phi_{s,ra}$ with MDT coupled with regression analysis, Equation (41)**	**Deviation $\Delta\Phi_s$** and **(Error)**
0	0	0	
0.10	0.35683	0.35621	−0.00062 (−0.17%)
0.20	0.50517	0.50801	0.00284 (0.01%)
0.30 (Dimensionless threshold time)	0.63379	0.63410	0.00031 (0.05%)
			RMSE = 0.10%

Table 5. Numerical values of the dimensionless surface temperature $\Phi_s(\tau)$ for a long cylinder obtained with MDT coupled with regression analysis inside the sub−domain $0 < \tau \leq 0.20$.

Dimensionless time τ	**Exact analytical Φ_s, Equation (31)**	**Approximate semi analytical $\Phi_{s,ra}$ with MDT coupled with regression analysis, Equation (42)**	**Deviation $\Delta\Phi_s$ (Error)**
0	0	0	
0.05	0.28104	0.53548	0.00056 (0.10%)
0.10	0.41833	0.53548	0.00056 (0.10%)
0.15	0.53492	0.53548	0.00056 (0.10%)
0.20 (Dimensionless threshold time)	0.64277	0.64274	− 0.00003 (−0.005%)
			RMSE = 0.08%

Table 6. Numerical values of the dimensionless surface temperature $\Phi_s(\tau)$ for a sphere obtained with MDT coupled with regression analysis inside the sub–domain $0 < \tau \leq 0.10$.

Dimensionless time τ	**Exact, analytical Φ_s from Equation (33)**	**Approximate semi analytical $\Phi_{s,ra}$ with MDT coupled with regression analysis, Equation (43)**	**Deviation $\Delta\Phi_s$ (error)**
0	0	0	
0.05	0.31217	0.31195	−0.00022 (0.07%)
0.10 (Dimensionless threshold time)	0.48676	0.48710	0.00034 (0.07%)

	RMSE = 0.07%

9.3. Asymptotic sub–domain for dimensionless time $\tau \rightarrow \infty$

(1) Plane wall

$$\Phi_{s,asy}(\tau) = \tau + \frac{1}{3}, \text{for } \tau > 0.3 \tag{44}$$

(2) Long cylinder

$$\Phi_{s,asy}(\tau) = 2\tau + \frac{1}{4}, \text{for } \tau > 0.2 \tag{45}$$

(3) Sphere

$$\Phi_{s,asy}(\tau) = 3\tau + \frac{1}{5}, \text{for } \tau > 0.1 \tag{46}$$

10. Conclusions

The present study dealt with the analysis of the unsteady heat conduction influenced by surface heat flux in large walls, long cylinders and spheres. The main conclusion that may drawn from the study is that the surface temperatures provided by the Method of Discretization in Time (DMT) coupled with regression analysis of the surface temperature deviations produce levels of exceptional accuracy in the problematic “small time” sub–domain $0 < \tau \leq \tau_{th} = 0.30$ for the large wall, the “small time” sub–domain $0 < \tau \leq \tau_{th} = 0.20$ for the long cylinder in the “small time” sub–domain $0 < \tau \leq \tau_{th} = 0.10$ for the sphere.

Additionally, the surface temperatures in the “large time” sub–domains $\tau \geq \tau_{th} = 0.30$ in the large wall, $\tau \geq \tau_{th} = 0.20$ in the long cylinder and $\tau \geq \tau_{th} = 0.10$ in the sphere can be properly handled with the three standard asymptotic **Equations** (**32**)–(**34**) for the large plate, the long cylinder and the sphere, respectively.

Nomenclature

c_v	specific heat capacity at constant volume
k	thermal conductivity
n	geometric parameter
q_S	uniform surface heat flux
L_c	characteristic length
t	time
t_{eq}	equivalent time, $\frac{L_c^2}{\alpha}$
t_{th}	threshold time
T	temperature
T_c	center temperature
T_{eq}	equivalent temperature, $\frac{q_s L_c}{k}$
T_{in}	initial temperature
T_m	mean temperature
T_s	surface temperature
x	space coordinate
X	dimensionless space coordinate, $\frac{x}{L_c}$

Greek letters

α	thermal diffusivity, $\frac{k}{\rho c_v}$
τ	dimensionless time or Fourier number, $\frac{t}{t_{eq}}$
τ_{th}	dimensionless threshold time, $\frac{t_{th}}{t_{eq}}$
Φ	dimensionless temperature, $\frac{T - T_{in}}{T_{eq}}$
ρ	density
Subscripts	
asy	asymptotic
ra	regression analysis
1	first time jump, transversal line 1
2	second time jump, transversal line 2

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Appendix. Exact, analytical temperature distributions

The exact, analytical temperature distributions expressed in dimensionless form for the three simple bodies are given by the following infinite series in Luikov [3]:

(1) Large wall

$$\Phi(X,\tau) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right) - 2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{\mu_n^2}\right]\cos(\mu_n X)\ \exp(-\mu_n^2\,\tau) \qquad \text{(A1a)}$$

in conjunction with the eigenvalues

$$\mu_n = n\pi;\ \ n = 1, 2, 3, \ldots. \qquad \text{(A1b)}$$

(2) Long cylinder

$$\Phi(X,\tau) = 2\tau + \left(\frac{X^2}{2} - \frac{1}{4}\right) - 2\sum_{n=1}^{\infty}\left[\frac{1}{\mu_n^2 J_0(\mu_n)}\right]J_0(\mu_n X)\exp(-\mu_n^2\,\tau) \qquad \text{(A2a)}$$

Herein, the eigenvalues μ_n are the positive roots of the transcendental equation

$$J_1(\mu_n) = 0, \quad n = 1, 2, 3, ... \tag{A2b}$$

where $J_1(*)$ is the Bessel function of the first kind and order one.

(3) Sphere

$$\Phi(X,\tau) = 3\tau + \left(\frac{X^2}{2} - \frac{3}{10}\right) - 2\sum_{n=1}^{\infty} \left[\frac{1}{\mu_n^2 \cos(\mu_n)}\right] \frac{\sin(\mu_n X)}{\mu_n X} \exp\left(-\mu_n^2\,\tau\right) \tag{A3a}$$

Herein, the eigenvalues μ_n are the positive roots of the transcendental equation

$$\mu_n = \tan \mu_n; \quad n = 1, 2, 3, ... (\tag{A3b}$$

A method of accelerating the convergence of the series in **Equations** (**A1a**), (**A2a**) and (**A3a**) has been developed by Mikhailov and M. N. Özişik [16].

Approximate analytical solutions for "very large time"

The common exponential function with negative afrguments $\exp\left(-\mu_n^2\,\tau\right); n = 1,2,3$ ... appearing in **Equations** (**A1a**), (**A2a**) and (**A3a**) decays rapidly with the dimensionless time τ. Thre particular situation is tied up to "very large" dimensionless time $\tau \to \infty$ wherein the subsequent $\exp\left(-\mu_n^2\,\tau\right); n = 1,2,3$ ... vanishes. This signifies that the infinite series in **Equations** (**A1a**), (**A2a**) and (**A3a**) reduce to the corresponding asymptotic algebraic equations depending on τ linearly and depending on *X* parabolically.

(1) Large wall

$$\Phi_{asy}(X,\tau) = \tau + \left(\frac{X^2}{2} - \frac{1}{6}\right) \tag{A4}$$

(2) Long cylinder

$$\Phi_{asy}(X,\tau) = 2\tau + \left(\frac{X^2}{2} - \frac{1}{4}\right) \tag{A5}$$

(3) Sphere

$$\Phi_{asy}(X,\tau) = 3\tau + \left(\frac{X^2}{2} - \frac{3}{10}\right) \tag{A6}$$